PARAMODULAR CONJECTURE FOR GENUS 2 CURVES
RALF SCHMIDT

The paramodular conjecture states the following.

**Conjecture** (Brumer–Kramer). Let $A$ be an abelian surface over $\mathbb{Q}$ of conductor $N$ such that $\text{End}_{\mathbb{Q}}(A) = \mathbb{Z}$. Then there exists a cuspidal Siegel paramodular form $f \in S_{2}^{(2)}(\Gamma_{\text{para}}(N))$, a newform and eigenform of degree 2, weight 2, and level $N$, such that

$$L(A, s) = L(f, s).$$

If $\text{End}_{\mathbb{Q}}(A)$, the ring of endomorphisms of $A$ defined over $\mathbb{Q}$, is larger than $\mathbb{Z}$, then there is still a Siegel paramodular form, but it is not necessarily a cusp form.

The goal of this lecture is to explain why this conjecture is natural from the point of view of automorphic representations.

1. **Elliptic curve case**

As motivation, we first turn to the case of elliptic curves.

We first work locally. Let $F$ be a nonarchimedean local field with char $F = 0$, for example $F = \mathbb{Q}_{p}$. Let $E$ be an elliptic curve over $F$ of conductor $N$. We choose an auxiliary prime $\ell$ different from the residue characteristic of $F$. We form the Tate module $T_{\ell}(E)$, a $\mathbb{Z}_{\ell}$-module of rank 2 equipped with an action of $G(F/F)$; tensoring with $\mathbb{Q}_{\ell}$, we obtain a continuous representation $G(F/F) \to \text{GL}(2, \mathbb{Q}_{\ell})$.

There is a procedure to convert this Galois representation into a complex-valued representation, due to Grothendieck and Deligne. One can almost think of this as choosing an embedding $\mathbb{Q}_{\ell} \hookrightarrow \mathbb{C}$; however, one must fix it up topologically to ensure that the representation is continuous. In any event, we arrive at a representation $W'(\overline{F}/F) \to \text{GL}(2, \mathbb{C})$ where $W'(\overline{F}/F)$ is the Weil–Deligne group; and then this representation is independent of the choice of $\ell$. (For a nice reference on the Weil–Deligne group and the concomitant representation theory, see Rohrlich 1994).

We pause to give a brief description of the Weil–Deligne group $W'(\overline{F}/F)$. Let $\mathcal{O}$ be the valuation ring of $F$, let $\mathfrak{p} = (\varpi) \subseteq \mathcal{O}$ be the maximal ideal, let $k = \mathcal{O}/\mathfrak{p}$, and let $q = |k|$. Then we have an exact sequence

$$1 \to I \to G(\overline{F}/F) \to G(k/k) \to 1$$

where $I$ is the inertia group whose fixed field in $\overline{F}$ is the maximal unramified extension of $F$. Inside $G(k/k)$ is the geometric Frobenius automorphism $\text{Frob}$ with $\text{Frob}^{-1}(x) = x^{q}$, and $G(k/k)$ is topologically generated by $\text{Frob}$. Choose a lift $\Phi \in W(\overline{F}/F)$ of $\text{Frob}$, and let

$$W(\overline{F}/F) = \bigsqcup_{n \in \mathbb{Z}} I\Phi^{n}$$

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and

\[ W'(\overline{F}/F) = W(\overline{F}/F) \times \mathbb{C} \]

where \(gzg^{-1} = \omega(g)z\) where \(\omega : W(\overline{F}/F) \to \mathbb{C}\) has \(I \subseteq \ker \omega\) and \(\omega(\Phi) = q^{-1}\).

Now, with our Weil–Deligne representation \(W'(\overline{F}/F) \to \text{GL}(2, \mathbb{C})\) in hand, we pass to an automorphic representation. This is the content of the Local Langlands correspondence, which was proven for \(\text{GL}(2)\) in the 1980s, and we obtain an irreducible admissible representation \(\pi : \text{GL}(2, F) \to V\) into a \(\mathbb{C}\)-vector space (in general, infinite dimensional). The local Langlands correspondence for \(\text{GL}(1)\) is the content of local class field theory: a character of \(\mathbb{A}_F\) has \(\pi\) as even \(\mathfrak{p}\)-adic representation. By strong approximation, \(\Phi\) determines a map on \(\text{GL}(2, \mathbb{C})\) — in this isomorphism, we have the inverse Frobenius \(\Phi\) mapping to a uniformizer \(\varpi\). In any event, we obtain a correspondence between characters of \(W'(\overline{F}/F)\) and irreducible admissible representations of \(F^\times = \text{GL}(1, F)\). The local Langlands correspondence is a generalization of this from \(\text{GL}(1, F)\) to \(\text{GL}(2, F)\), in our setting. (The case \(\text{GL}(n, F)\) has been known since around 2005.)

From the representation \(\pi\) of \(\text{GL}(2, F)\), we find a local newform \(v \in V\); this local newform \(v\) is invariant under \(\Gamma_0(n)\) where \(n \in \mathbb{Z}_{\geq 0}\) is the conductor exponent of \(E\) over \(F\). We have \(V^\Gamma_0(0) \subset V^\Gamma_0(1) \subset \ldots\) and eventually some \(V^\Gamma_0(n)\) is nonzero; the dimensions are \(0, 0, \ldots, 0, 1, 2, 3, \) and the \(1\) occurs at the \(n\)th step: the analytic and arithmetic conductors agree. (The larger dimensional spaces reflect the theory of oldforms, due to Atkin–Lehner.)

Having formulated this local recipe, taking an elliptic curve over a local field and producing a local newform (a line in \(V\)), we turn to the global modularity statement. Let \(E\) be an elliptic curve over \(\mathbb{Q}\). By localizing, we obtain a collection of irreducible admissible representations \(\pi_p\) of \(\text{GL}(2, \mathbb{Q}_p)\), including \(p = \infty\). These yield a single representation \(\pi = \bigotimes_p \pi_p\), an irreducible admissible representation of \(\text{GL}(2, \mathbb{A})\). The important thing: this representation is automorphic!

We recover a classical modular form from the local newform as follows. From \(\pi = \bigotimes_p \pi_p\), we obtain \(v_p \in \pi_p\) the local newform for \(p < \infty\) (defined up to scalar) and a weight vector for \(p = \infty\). This yields \(\bigotimes_p v_p \in \bigotimes_p \pi_p = \pi\). Such a vector yields an automorphic form \(\Phi\) that is left \(\text{GL}(2, \mathbb{A})\)-invariant. By strong approximation, \(\Phi\) determines a map on \(\text{GL}(2, \mathbb{R})^+\) as \(f \mapsto \Phi(g);\) the map \(f : \mathfrak{H} \to \mathbb{C}\) is a classical modular form \(f \in S_2(\Gamma_0(N))\) where \(n = \prod_p p^{\nu_p}\).

\section{2. Paramodular case}

We now consider the paramodular case in genus 2. We follow the same outline as in the previous section. (We also ignore many subtleties in an effort to convey the main idea.)

Let \(A\) be a principally polarized abelian surface over the local field \(F\). (One can relax this condition allowing certain bad primes, but one loses holomorphicity, something we must give up in higher genus anyway.) The Tate module \(T_\ell(A)\) has a symplectic structure, so we obtain a representation \(\pi : G(\overline{F}/F) \to \text{Sp}(4, \mathbb{Q}_\ell)\) after Tate twist, and then a Weil–Deligne representation. The local Langlands correspondence (proven in 2010 by Gan and Takeda), yields an irreducible admissible representation of \(\text{SO}(5, F) \simeq \text{PGSp}(4, F)\), which we can think of as a representation of \(\text{GSp}(4, F)\) with trivial central character.

Now let \(A\) be a principally polarized abelian surface over \(\mathbb{Q}\). Localizing and applying the previous paragraph, we obtain representations \(\pi_p\) of \(\text{GSp}(4, \mathbb{Q}_p)\) with trivial central character
for all $p$. Put together, these yield $\pi = \bigotimes'_p \pi_p$ an irreducible admissible representation of $\text{GSp}(4, \mathbb{A})$. The main content of the paramodular conjecture is that $\pi$ is an automorphic representation.

Assuming this is the case, we want to pick out local newforms in each $\pi_p$ giving a global newform. We define the paramodular group $\Gamma_{\text{para}}(n)$ or $\Gamma_{\text{para}}(p^n)$ locally to be the subgroup of $\text{GSp}(4, F)$ belonging to

$$\Gamma(n) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} & p^{-n} \\ p^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ p^n & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ p^n & p^n & p^n & \mathcal{O} \end{pmatrix}.$$  

Why this group? Because it is the one that admits a nice local newform theory like $\text{GL}(2)$: the first nonzero fixed subspace has dimension 1, spanned by a $v_p$ for all $p$. If we try the analogue of $\Gamma_0(N)$ (lower $2 \times 2$ matrix divisible by $N$) then the numerology does not match up: there is no uniqueness, and no agreement of analytic and algebraic conductors.

Then, from these local newforms we form $v = \bigotimes_p v_p$, and obtain an automorphic form $\Phi : \text{GSp}(4, \mathbb{A}) \to \mathbb{C}$ left invariant under $\text{GSp}(4, \mathbb{Q})$. Strong approximation again says that $\Phi$ is determined by a function $\text{GSp}(4, \mathbb{R})^+$, and this is our (classical) Siegel modular form of weight 2 on the paramodular group $\Gamma_{\text{para}}(N)$.

In general, from a principally polarized abelian variety of dimension $g$ we would land in $\text{Sp}(2n, \mathbb{C})$; the local Langlands correspondence is now a theorem of Arthur, and we obtain an irreducible admissible representation of $\text{SO}(2n + 1, F)$ locally. Here, there is no longer any holomorphic structure on the associated symmetric space.